NEARLY HOLOMORPHIC FUNCTIONS AND RELATIVE DISCRETE SERIES OF WEIGHTED L^2-SPACES ON BOUNDED SYMMETRIC DOMAINS

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ABSTRACT. Let $\Omega = G/K$ be a bounded symmetric domain in a complex vector space V with the Lebesgue measure $dm(z)$ and the Bergman reproducing kernel $h(z,w)^{-p}$. Let $d\mu_\alpha(z) = h(z,\bar{z})^\alpha dm(z)$, $\alpha > -1$, be the weighted measure on Ω. The group G acts unitarily on the space $L^2(\Omega, \mu_\alpha)$ via change of variables together with a multiplier. We consider the discrete parts, also called the relative discrete series, in the irreducible decomposition of the L^2-space. Let $\bar{D} = B(z,\bar{z})\partial$ be the invariant Cauchy-Riemann operator. We realize the relative discrete series as the kernels of the power $\bar{D}^{m+1}$ of the invariant Cauchy-Riemann operator $\bar{D}$ and thus as nearly holomorphic functions in the sense of Shimura. We prove that, roughly speaking, the operators $\bar{D}^m$ are intertwining operators from the relative discrete series into the standard modules of holomorphic discrete series (as Bergman spaces of vector-valued holomorphic functions on Ω).

1. INTRODUCTION

Let Ω be a bounded symmetric domain in a complex vector space V with the Lebesgue measure $dm(z)$. The Bergman reproducing kernel is up to a constant $h(z,\bar{w})^{-p}$, where $h(z,\bar{w})$ is an irreducible polynomial holomorphic in z and antiholomorphic in w. We consider the weighted measure $d\mu_\alpha(z) = h(z,\bar{z})^\alpha dm(z)$ for $\alpha > -1$ and corresponding L^2-space $L^2(\Omega, \mu_\alpha)$ on Ω. The group G of biholomorphic mappings of Ω acts unitarily on the L^2-space via change of variables together with a multiplier, and the weighted Bergman space is then an irreducible invariant subspace. The irreducible decomposition of the L^2-space under the G-action has been given by Shimeno [11]. It is proved there abstractly (via identifying the infinitesimal characters) that all the discrete parts (called relative discrete series) appearing in the decomposition are holomorphic discrete series. In this paper we consider their explicit realization.

To illustrate our main results we consider the case of the unit disk. The Bergman reproducing kernel is $(1 - z\bar{w})^{-2}$, and the weighted measure in question is $d\mu_\alpha(z) = (1-|z|^2)^\alpha dm(z)$. The group $G = SU(1,1)$ acts unitarily on $L^2(D, \mu_\alpha)$ via a projective

representation

$$\pi_\nu(g)f(z) = f(g^{-1}z)(cz+d)^{-\nu}, \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $\nu = \alpha + 2$. To study the relative discrete series we introduce the invariant Cauchy-Riemann operator $\bar{D} = (1-|z|^2)^2\bar\partial$. The operator $\bar{D}$ intertwines the action π_ν with the action $\pi_{\nu-2}$, which can be proved by direct calculation. The kernel ker $\bar{D}$ of $\bar{D}$ on the weighted L^2-space is the weighted Bergman space $L^2_a(\Omega, \mu_\alpha)$ of holomorphic functions, which gives one of the relative discrete series. It is naturally to expect that the kernel ker $\bar{D}^{m+1}$ of the iterate of $\bar{D}$ will give us the other relative discrete series. The functions that are in the kernel ker D^{m+1} can be written as polynomial of $q(z) = \frac{\bar{z}}{1-|z|^2}$ of degree $\leq m$ with coefficients being holomorphic functions. Those functions, following Shimura, are called *nearly holomorphic functions*. The function $q(z)$ actually is the holomorphic differential of the Kähler potential $\log(1-|z|^2)^{-2}$. Indeed $q(z) = \frac{1}{2}\partial_z \log(1-|z|^2)^{-2}$. Moreover it has a Jordan theoretic meaning as the *quasi-inverse* of $\bar{z}$ with respect to z in $\bar{\mathbb{C}}$ with the Jordan triple product $\{\bar{u}z\bar{v}\} = 2\bar{u}z\bar{v}$. The key result is that each power $q(z)^m = \frac{\bar{z}^m}{(1-|z|^2)^m}$, for $0 \leq m < \frac{\alpha+1}{2}$ generates a relative discrete series. Denote corresponding the relative discrete series by $A^{2,\alpha}_m$. Then the operator $\bar{D}^m$ is an intertwining operator from $A^{2,\alpha}_m$ into the weighted Bergman space in $L^2(\Omega, \mu_{\alpha-2m})$, namely $L^2_a(\Omega, \mu_{a-2m})$. Moreover all relative discrete series are obtained in this way.

When $\Omega = G/K$ is a general bounded symmetric domain the corresponding function $q(z)$, defined as the differential of the Kähler potential, can indeed be expressed in term of quasi-inverse in the Jordan triple V. See Proposition 3.1. Let $\bar{D}$ be the invariant Cauchy Riemann operator. Then it is proved in [9] that the iterate $\bar{D}^m$ maps a function on Ω to a function with value in the symmetric subtensor space $S_m(V)$ of $\otimes^m V$. Decompose $S_m(V)$ into irreducible subspaces under K. Let $\underline{\mathbf{m}}$ be the signature of an irreducible subspace and $\bar\Delta_{\underline{\mathbf{m}}}$ the highest weight vector in that space, considered as a polynomial function on V'. Now the function $q(z)$ is a V'-valued function on Ω, thus $\bar\Delta_{\underline{\mathbf{m}}}(q(z))$ is a scalar-valued function on Ω. We prove that $\bar\Delta_{\underline{\mathbf{m}}}(q(z))$ is in the space $L^2(\Omega, \mu_\alpha)$ when $\underline{\mathbf{m}}$ satisfies certain condition; see Proposition 4.1. We further prove that it generates an irreducible subspace, namely a relative discrete series, and is the highest weight vector, and that the operators $\bar{D}^m$ are intertwining operators from the relative discrete series onto the weighted Bergman space of holomorphic functions with values in the irreducible subspace $S_{\underline{\mathbf{m}}}(V)$ of the symmetric tensor $S_m(V)$, the later being a standard module of holomorphic discrete series. We thus realize the relative discrete series in the kernel of the power $\bar{D}^{m+1}$ and as nearly holomorphic functions in the sense of Shimura ([12] and [13]).

Finally in the last section we consider as an example the unit ball in $\mathbb{C}^n$. We calculate directly, via the adjoint operator $\bar{D}^*$, the highest weight vector in the relative

discrete series. The realization of the relative discrete series has also been studied in [9].

Our results explain geometrically why the relative discrete series are equivalent to the weighted Bergman spaces with values in *symmetric tensor space* of the tangent space. Moreover, since the highest weight vectors are quite explicitly given we understand better the analytic nature of the functions in the discrete series. We hope that our result will be helpful in understanding the L^p-spectral properties of the irreducible decomposition, for example, the L^p-boundedness of the orthogonal projection into the relative discrete series.

2. Invariant Cauchy-Riemann Operator $\bar{D}$ and Nearly Holomorphic Functions on Kähler manifolds

We recall in this section briefly some preliminary results on invariant Cauchy-Riemann operators and nearly holomorphic functions on Kähler manifolds; see [12], [2], [14], and [15].

Let Ω be a Kähler manifold with the Kähler metric locally given by the matrix $(h_{i\bar{j}})$, with $h_{i\bar{j}} = \frac{\partial^2 \Psi}{\partial z_i \partial \bar{z}_j}$ and a potential Ψ. Let $T^{(1,0)}$ be its holomorphic tangent bundle. Let W be a Hermitian vector bundle over Ω, and $C^\infty(\Omega, W)$ its smooth sections. The invariant Cauchy-Riemann operator $\bar{D}$ locally defined as follows. If $f = \sum_\alpha f_\alpha e_\alpha$ is any section of W, then

$$\bar{D}f = \sum_{\alpha, i, j} h^{\bar{j}i} \frac{\partial f_\alpha}{\partial \bar{z}^j} \partial_i \otimes e_\alpha.$$

It maps $f \in C^\infty(\Omega, W)$ to $\bar{D}f \in C^\infty(\Omega, T^{(1,0)} \otimes W)$. Denote $S_m(T^{(1,0)})$ the symmetric tensor subbundle of $\otimes^m T^{(1,0)}$. We recall some known properties of the operator $\bar{D}$. See [9].

Lemma 2.1. *The following assertions hold.*

(1): *The operator $\bar{D}$ is an intertwining operator: If g is a biholomorphic mapping of Ω, then*

$$(2.1) \qquad\qquad \bar{D}(g_W f) = ((dg)^{-1} \otimes g_W)Df,$$

where g_W is the induced action of g on sections of W and $dg(z) \colon T_z^{(1,0)} \mapsto T_{gz}^{(1,0)}$ is the differential of g.

(2): *The iterate $\bar{D}^m$ of $\bar{D}$ maps $C^\infty(\Omega, W)$ to $C^\infty(\Omega, W \otimes S_m(T^{(1,0)}))$.*

For our later purpose we can assume that Ω is some domain in a vector space V with coordinates $\{z_j\}$, and that all the bundles are trivial. The space $T_z^{(1,0)}$ will be

identified with V. So let W be a vector space and we will consider the $C^\infty(\Omega, W)$ of W-valued C^∞-functions on Ω.

Let

$$q(z) = \partial \Psi = \sum_j \frac{\partial \Psi}{\partial z_j} dz_j.$$

Here Ψ is the Kähler potential, and $\{dz_j\}$ is the dual basis for the holomorphic cotangent space V'. Thus $q(z)$ is a function with values in V'. Following Shimura [12] we call a W-valued function $f \in C^\infty(\Omega, W)$ nearly holomorphic if f is a polynomial of $q(z)$ with holomorphic coefficients. We denote $\mathcal{N}_m$ the space of scalar-valued nearly holomorphic functions that are polynomial of degree $\leq m$, namely those functions $f(z) = \sum_{|\underline{\beta}| \leq m} c_{\underline{\beta}}(z) q(z)^{\underline{\beta}}$ where $c_{\underline{\beta}}(z)$ are holomorphic functions.

We denote Id the identity tensor in the tensor product $V \otimes V'$. By the direct calculation we have

(2.2)
$$\bar{D} q(z) = \text{Id};$$

see [14]. We generalize this formula as follows; the proof of it is quite straightforward and we omit it.

Lemma 2.2. *We have the following differentiation formula*

(2.3)
$$\bar{D}^m(\otimes^m q(z)) = m! Id,$$

where Id in the right hand denotes the identity tensor in the tensor product $(S_m V) \otimes S_m(V') = (S_m V) \otimes (S_m V)'$.

Remark 2.3. The formula (2.2) was observed earlier by Shimura [12] and Peetre [8]; in the later paper explicit formulas were given for the Laplace operators on weighted L^2-spaces on bounded symmetric domains, where the function $q(z)$ also appears.

Example 2.4. We consider the case of the unit disk. The operator $\bar{D} = (1 - |z|^2)^2 \bar{\partial}$ The function $q(z)$ is $\frac{\bar{z}}{1-|z|^2}$ (or exactly it is $\frac{\bar{z}}{1-|z|^2} dz$). The above formula amounts to

$$\bar{D}^m \left(\frac{\bar{z}}{1 - |z|^2} \right)^m = m!,$$

which can be proved by direct calculations. It can also be proved by using the formula

$$\bar{D}^m = (1 - |z|^2)^{m+1} \left(\frac{\partial}{\partial \bar{z}} \right)^m (1 - |z|^2)^{m-1},$$

see [14]. Indeed,

$$\bar{D}^m(\frac{\bar{z}}{1-|z|^2})^m$$

$$= (1-|z|^2)^{m+1}(\frac{\partial}{\bar{\partial}z})^m(1-|z|^2)^{m-1}(\frac{\bar{z}}{1-|z|^2})^m$$

$$= (1-|z|^2)^{m+1}(\frac{\partial}{\bar{\partial}z})^m\frac{\bar{z}^m}{1-|z|^2}$$

$$= (1-|z|^2)^{m+1}\sum_{l=0}^m \binom{m}{l}m(m-1)\cdots(m-l+1)\bar{z}^{m-l}\frac{(m-l)!z^{m-l}}{(1-|z|^2)^{m-l+1}}$$

$$= m!\sum_{l=0}^m \binom{m}{l}(z\bar{z})^{m-l}(1-|z|^2)^l$$

$$= m!$$

The calculations are somewhat combinatorially intriguing.

Using the above result we get immediately the following characterization of nearly holomorphic functions. This is proved in [12], Proposition 2.4, for classical domains. It can be proved for all Kähler manifolds via the same methods.

Lemma 2.5. *Consider the operator $\bar{D}^{m+1}$ on the space $C^\infty(D)$ of C^∞-functions on D. Then*

$$Ker\,\bar{D}^{m+1} = \mathcal{N}_m.$$

We recall the identification of polynomial functions with symmetric tensors. This will clarify conceptually some calculations in the next section. There is a pairing

$$(\phi, \psi) \in S_m(V) \times S_m(V') \mapsto [\phi, \psi] \in \mathbb{C},$$

between the symmetric tensor spaces $S_m(V)$ and $S_m(V')$, via the natural pairing between $\otimes^m V$ and $\otimes^m V'$. Now for each element ϕ in the symmetric tensor space $S_m(V)$ there corresponds a homogeneous polynomial function of degree m on the space V', also denoted by ϕ, such that

(2.4) $$[\phi, v' \otimes v' \otimes \cdots v'] = \phi(v')$$

for any $v' \in V'$.

Using this convention we see that a function $f \in C^\infty(\Omega)$ is in $\mathcal{N}_m$ if and only if there exist holomorphic functions g_k with values in the tensor product $S_k(V)$, $k = 0, 1, \ldots, m$, such that

(2.5) $$f(z) = \sum_{k=0}^m g_k(q(z)).$$

3. Nearly Holomorphic Functions on Bounded Symmetric Domains

In this section we assume that $\Omega = G/K$ is a bounded symmetric domain of rank r in a complex vector space V. Here G is the identity component of the group of biholomorphic mappings of Ω and K is the isotropy group at $0 \in V$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. The space V has a Jordan triple structure so that the space $\mathfrak{p}$ is explicitly described; see [6], whose notation and results will be incorporated here. So let $Q(z) : \bar{V} \to V$ be the quadratic operator. The

$$\mathfrak{p} = \{\xi_v = v - Q(z)\bar{v}\}$$

viewed as holomorphic vector fields on Ω. Let $D(z, \bar{v})w = \{z\bar{v}w\} = (Q(z+w) - Q(z) - Q(w))\bar{v}$ be the Jordan triple product. We normalize the K-invariant Hermitian inner product $\langle z, w \rangle$ on V so that a minimal tripotent has norm 1. This can also be calculated by

$$(3.1) \qquad \langle z, w \rangle = \frac{1}{p} \operatorname{Tr} D(z, \bar{w})$$

where p is an integer called the genus of Ω. We identify then the vector space V' with $\bar{V}$ via this scalar product.

Let $dm(z)$ be the corresponding Lebesgue measure on V. The Bergman reproducing kernel on D is the $ch(z, w)^{-p}$ for some positive constant c. Let

$$B(z, \bar{w}) = I - D(z, \bar{w}) + Q(z)Q(\bar{w})$$

the Bergman operator. $B(z, \bar{w})$ is holomorphic in the first argument and anti-holomorphic in the second. (We write $B(z, \bar{w})$ instead of $B(z, w)$ as in [6] in order to differ it from $B(\bar{z}, w)$ which is acting on the space $\bar{V}$.) The Bergman metric at $z \in \Omega$ defined by the metric $\partial_j \bar{\partial}_k \log h(z, \bar{z})^{-p}$ on Ω is then

$$p\langle B(z, \bar{z})^{-1}z, w \rangle;$$

and

$$\det B(z, \bar{z}) = h(z, \bar{z})^p$$

See [6]. For some computational convenience we will choose and fix the metric on Ω to be

$$(3.2) \qquad \boxed{\langle B(z, \bar{z})^{-1}z, w \rangle}.$$

The invariant Cauchy-Riemann operator is

$$\bar{D} = B(z, \bar{z})\bar{\partial},$$

and the N-function defined in the previous section is now (with a normalizing constant)

$$(3.3) \qquad \boxed{q(z) = \frac{1}{p}\partial \log \det B(z, \bar{z})^{-1}}$$

We shall find an explicit formula for the function $q(z)$ on Ω. Recall first the notion of *quasi-inverse* in the Jordan triple V; see [6]. Let $z \in V$ and $\bar{w} \in \bar{V}$. The element z is called quasi-invertible with respect to w if $B(z, \bar{w})$ is invertible and its quasi-inverse is given by

$$z^{\bar{w}} = B(z, \bar{w})^{-1}(z - Q(z)\bar{w}).$$

Similarly we define the quasi-inverse of an element $\bar{z} \in \bar{V}$ with respect to $w \in V$.

Proposition 3.1. *The function $q(z)$ on Ω is given by*

$$q(z) = \bar{z}^z = B(\bar{z}, z)^{-1}(\bar{z} - Q(\bar{z})z)$$

Proof. For some computational convenience we consider, instead of the holomorphic differential in (3.3), the anti-holomorphic differential

$$\frac{1}{p}\bar{\partial} \log \det B(z, \bar{z})^{-1} = -\frac{1}{p}\bar{\partial} \log \det B(z, \bar{z}).$$

Let $\bar{v} \in \bar{V}$. By the definition of B-operator we have

$$
\begin{aligned}
& B(z, \bar{z} + t\bar{v}) \\
(3.4) \quad & = 1 - D(z, \bar{z} + t\bar{v}) + Q(z)Q(\bar{z} + t\bar{v}) \\
& = 1 - D(z, \bar{z}) + Q(z)Q(\bar{z}) + t(-D(z, \bar{v}) + Q(z)Q(\bar{z}, \bar{v}) + t^2 Q(z)Q(\bar{v}) \\
& = B(z, \bar{z})\left(I + tB(z, \bar{z})^{-1}(-D(z, \bar{v}) + Q(\bar{z}, \bar{v})) + t^2 B(z, \bar{z})^{-1}Q(z)Q(\bar{v})\right).
\end{aligned}
$$

Thus the first order term in t in $\log \det B(z, \bar{z} + t\bar{v})$ is

$$(3.5) \qquad \mathrm{Tr}(B(z, z)^{-1}(-D(z, \bar{v}) + Q(z)Q(\bar{z}, \bar{v})).$$

We recall a formula in [6] (see (JP30))

$$B(z, \bar{z})D(z^{\bar{z}}, v) = D(z, \bar{v}) - Q(z)Q(\bar{z}, \bar{v}).$$

Therefore (3.5) is

$$- \mathrm{Tr}\, D(z^{\bar{z}}, v) = -p\langle z^{\bar{z}}, v\rangle$$

by the formula (3.1). Summarizing we find

$$\frac{1}{p}\bar{\partial}_v \log \det B(z, \bar{z})^{-1} = \langle z^{\bar{z}}, v\rangle,$$

which is the desired formula. □

Now the group K acts on Ω and keeps the function $h(z, \bar{z})$-invariant. Thus we get, in view of the formula (3.3),

$$(3.6) \qquad\qquad q(kz) = (k^{-1})'q(z),$$

where $(k^{-1})'$ on $q(z) \in V'$ is the dual of k^{-1} on V.

In particular, since the function $q(z)$ is a V'-valued function on Ω, we have, for any homogeneous polynomial function f on V', a scalar-valued function $f(q(z))$. The following lemma then follows from (3.6) and the K-invariance of the pairing between $S_m(V')$ and $S_m(V)$.

Lemma 3.2. *The map*

$$v \in S_m(V) \mapsto v(q(z)) = [v, \otimes^m q(z)]$$

is an invertible K-intertwining operator between the K-action on $S_m(V)$ and its regular action on functions on Ω.

We recall now the decomposition of $S_m(V)$ under K. To state the result we fix some notation. The complexification $\mathfrak{g}^{\mathbb{C}}$ of the Lie algebra $\mathfrak{g}$ has a decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^+ + \mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$, with $\mathfrak{k}^{\mathbb{C}}$ the complexification of the Lie algebra $\mathfrak{k}$ of K and $\mathfrak{p}^+ = V$. Let $\{e_1, \ldots, e_r\}$ be a frame of tripotents in V. Fix an Cartan subalgebra of of $\mathfrak{k}^{\mathbb{C}}$, and let $\gamma_1 > \cdots > \gamma_r$ be the Harish-Chandra strongly roots so that $e_1, \ldots, e_r$ are the corresponding root vectors. The ordering of the roots of $\mathfrak{g}^{\mathbb{C}}$ is so that $\mathfrak{p}^+$ is the sum of positive non-compact root vectors. We shall then speak of *highest weight modules* of $\mathfrak{g}^{\mathbb{C}}$ with respect to this ordering.

Lemma 3.3. *([4], [10] and [3]) The space $S_m(V)$ (respectively $S_m(V')$) under K is decomposed into irreducible subspaces with multiplicity one as*

$$S_m(V) = \sum_{\underline{\mathbf{m}}} S_{\underline{\mathbf{m}}}(V), \qquad (resp.\ S_m(V') = \sum_{\underline{\mathbf{m}}} S_{\underline{\mathbf{m}}}(V'))$$

where each $S_{\underline{\mathbf{m}}}(V)$ (resp. $S_{\underline{\mathbf{m}}}(V')$) is of highest weight $\underline{\mathbf{m}} = m_1\gamma_1 + \cdots + m_r\gamma_r$ (resp. lowest weight $-(m_1\gamma_1 + \cdots + m_r\gamma_r)$) with $m_1 \geq m_2 \geq \cdots \geq m_r \geq 0$, and the summation is over all $\underline{\mathbf{m}}$ with $|\underline{\mathbf{m}}| = m_1 + m_2 + \cdots + m_r = m$.

The highest weight vectors of $S_{\underline{\mathbf{m}}}(V)$ (respectively lowest weight vectors of $S_{\underline{\mathbf{m}}}(V')$) have constructed explicitly; see [3] and reference therein. Let Δ_j be the lowest weight vector of the fundamental representation $\underline{\mathbf{m}} = \underline{\mathbf{1}}^j = \gamma_1 + \cdots + \gamma_j$, $j = 1, \ldots, r$. The polynomial $\Delta = \Delta_r$ is the determinant function of the Jordan triple V. Then the lowest weight vector of $S_{\underline{\mathbf{m}}}(V')$ is

$$(3.7) \qquad \Delta_{\underline{\mathbf{m}}}(v) = \Delta_1(v)^{m_1-m_2} \cdots \Delta_{r-1}(v)^{m_{r-1}-m_r} \Delta_r(v)^{m_r},$$

viewed as polynomial of $v \in V$. Via the natural pairing between $S_{\underline{\mathbf{m}}}(V')$ and $S_{\underline{\mathbf{m}}}(V)$ we find that the highest weight vector of $S_{\underline{\mathbf{m}}}(V)$ is $\bar{\Delta}_{\underline{\mathbf{m}}}$ and

$$(3.8) \qquad \bar{\Delta}_{\underline{\mathbf{m}}}(w) = \bar{\Delta}_1(w)^{m_1-m_2} \cdots \bar{\Delta}_{r-1}(w)^{m_{r-1}-m_r} \bar{\Delta}_r(w)^{m_r},$$

viewed as polynomial of $w \in V' = \bar{V}$.

4. THE RELATIVE DISCRETE SERIES OF $L^2(\Omega, \mu_\alpha)$

In this section we find a family of relative discrete series by constructing some vectors that are in L^2-space and are highest weight vectors, namely annihilated by the positive vectors in $\mathfrak{g}^{\mathbb{C}}$ via the induced action of (4.1) (see below).

Let $\alpha > -1$. and consider the weighted measure

$$d\mu_\alpha = h(z,z)^\alpha dm(z).$$

The group G acts unitarily on the space $L^2(\Omega, d\mu_\alpha)$ via

$$(4.1) \qquad \pi_\nu(g)f(z) = f(g^{-1}z)J_{g^{-1}}(z)^{\frac{\nu}{p}}, \quad g \in G.$$

where $\nu = \alpha + p$ and J_g is the Jacobian determinant of g. We denote $L_a^2(\Omega, \mu_\alpha)$ the weighted Bergman space of holomorphic functions in $L^2(\Omega, \mu_\alpha)$.

We introduce now the weighted Bergman spaces of vector-valued holomorphic functions that will be used to realize the relative discrete series in $L^2(\Omega, d\mu_\alpha)$. Fix a signature $\underline{\mathbf{m}}$ with $m = m_1 + \cdots + m_r$. We denote $L_a^2(\Omega, S_{\underline{\mathbf{m}}}(V), \mu_\alpha)$ the weighted Bergman space of $S_{\underline{\mathbf{m}}}(V)$-valued holomorphic functions such that the following norm is finite

$$\|f\|^2 = \int_\Omega \langle (\otimes^m B(z, \bar{z})^{-1})f(z), f(z) \rangle \, d\mu_\alpha(z).$$

The group G acts unitarily on $L_a^2(\Omega, S_{\underline{\mathbf{m}}}(V), \mu_\alpha)$ via

$$(4.2) \qquad g \in G : f(z) \mapsto (J_{g^{-1}}(z))^{\frac{\nu}{p}} \otimes^m (dg^{-1}(z))^{-1} f(g^{-1}z).$$

This space is non trivial and forms an irreducible representation of G when $\underline{\mathbf{m}}$ satisfies the following condition:

$$(4.3) \qquad \frac{\alpha+1}{2} > m_1 \geq m_2 \geq \cdots \geq m_r \geq 0.$$

This follows directly from Theorem 6.6 in [5]; see also [11]. (We note here that non-triviality of the space can also be proved directly by expressing the inverse $B(z, \bar{z})^{-1}$ of the Bergman operator via the quasi-inverse developed in [6], quite similar to the proof of Proposition 4.1 below. However we will not go into the details here.)

Our first result is a construction of certain vectors in $L^2(\Omega, \mu_\alpha)$.

Proposition 4.1. *Suppose $\underline{\mathbf{m}}$ satisfies the condition 4.3. Then the functions $\bar{\Delta}_{\underline{\mathbf{m}}}(q(z))$ is in $L^2(\Omega, \mu_\alpha)$ and in $Ker\bar{D}^{m+1}$.*

We begin with fundamental representations $S_{\underline{\mathbf{m}}}(V)$ with signatures $\underline{\mathbf{m}} = \underline{\mathbf{1}}^j = \gamma_1 + \cdots + \gamma_j$ and highest weight vectors $\bar{\Delta}_j$, $j = 1, \ldots, r$.

Lemma 4.2. *Then the function $\Delta_j(q(z))$ is of the form*

$$(4.4) \qquad \bar{\Delta}_j(q(z)) = \frac{P(z, \bar{z})}{h(z, \bar{z})}$$

where $P(z, \bar{z})$ is a polynomial in $(z, \bar{z})$ of total degree not exceeding $2r$. In particular if $j = r$,

$$(4.5) \qquad \bar{\Delta}(q(z)) = \frac{\bar{\Delta}(z)}{h(z, \bar{z})}.$$

Proof. It follows from the Faraut-Koranyi expansion that

$$(4.6) \qquad h(v, \bar{w}) = \sum_{s=0}^{r} (-1)^r c_s K_{\mathbf{1}^s}(v, \bar{w}),$$

where $K_{\mathbf{m}}$ is the reproducing kernel of the subspace $\mathcal{P}^{\mathbf{m}}(V)$ of $\mathcal{P}(V)$ with signature $\mathbf{m}$ with the Fock-norm $\langle \cdot, \cdot \rangle_{\mathcal{F}}$ and c_s are positive constants; see [3]. Performing the inner product in the Fock space of the element $h(v, \bar{w})$ with the function $\Delta_{\mathbf{1}^j}(v)$ and using (4.6) we find that

$$\langle h(\cdot, \bar{w}), \Delta_j \rangle_{\mathcal{F}} = (-1)^s c_s \|\Delta_j\|_{\mathcal{F}}^2 \bar{\Delta}_j(\bar{w});$$

namely,

$$(4.7) \qquad \bar{\Delta}_j(\bar{w}) = \frac{1}{(-1)^s c_s} \|\Delta_j\|_{\mathcal{F}}^2 \langle h(\cdot, \bar{w}), \Delta_j \rangle_{\mathcal{F}}.$$

We take now $\bar{w} = \bar{z}^z$. Recall [6], Lemma 7.5, that

$$(4.8) \qquad h(v, \bar{z}^z) = \frac{h(v + z, \bar{z})}{h(z, \bar{z})}.$$

Substituting this into the previous formula we get

$$(4.9) \qquad \bar{\Delta}_j(\bar{z}^z) = \frac{1}{(-1)^s c_s h(z, \bar{z})} \|\Delta_j\|_{\mathcal{F}}^2 \langle h(\cdot + z, \bar{z}), \Delta_j \rangle_{\mathcal{F}}$$

Since $h(v + z, z)$ is a polynomial in z and $\bar{z}$ of degree $2r$, we see that $\Delta_j(\bar{z}^z)$ is of the declared form.

If $j = r$, we can then calculate $\langle h(\cdot, \bar{z}), \Delta_r \rangle_{\mathcal{F}}$ further. Expand $h(v + z, \bar{z})$ again using (4.6). We have

$$\langle h(\cdot + z, \bar{z}), \Delta_r \rangle_{\mathcal{F}} = \sum_{s=0}^{r} (-1)^s c_s \langle K_{\mathbf{1}^s}(\cdot + z, \bar{z}), \Delta_r \rangle_{\mathcal{F}}$$

(4.10)

$$= (-1)^r c_s \langle K_{\mathbf{1}^r}(\cdot + z, \bar{z}), \Delta_r \rangle_{\mathcal{F}},$$

because Δ_r is of degree r and it is orthogonal to those terms of lower degree. But

$$K_{\mathbf{1}^r}(z + v, z) = K_{\mathbf{1}^r}(v, z) + \dots$$

where the rest term is of lower order. Therefore by the same reason and by the reproducing property,

$$\langle h(z + \cdot, z), \Delta_r \rangle_{\mathcal{F}} = (-1)^r c_r \langle K_{\mathbf{1}^r}(\cdot, \bar{z}), \Delta_r \rangle_{\mathcal{F}} = (-1)^r c_r \|\Delta_r\|_{\mathcal{F}}^2 \overline{\Delta_r(z)}.$$

Substituting this into (4.9) we then get (4.5). $\qquad \square$

Remark 4.3. The norm $\|\Delta_{\mathbf{m}}\|_{\mathcal{F}}$ is calculated in [3], though we will not need it in the present paper.

Recall formula (3.8) for the highest weight vector $\bar{\Delta}_{\mathbf{m}}$. As a corollary we find immediately that

Corollary 4.4. *Then the function* $\bar{\Delta}_{\underline{\mathbf{m}}}(q(z))$ *is of the form*

$$\bar{\Delta}_{\underline{\mathbf{m}}}(q(z)) = \frac{P(z,\bar{z})}{h(z,\bar{z})^{m_1}}$$

where $P(z,\bar{z})$ *is a polynomial in* $(z,\bar{z})$.

We prove now the Proposition 4.1.

Proof. We estimate the norm of $\bar{\Delta}_{\underline{\mathbf{m}}}(q(z))$ in $L^2(\Omega, \mu_\alpha)$ by using the above Corollary. The polynomial $P(z,z)$ on Ω is bounded, say $|P(z,z)| \leq C$. We have

$$\int_\Omega |\frac{P(z,z)}{h(z,\bar{z})^{m_1}}|^2 d\mu_\alpha \leq C \int_\Omega h(z,\bar{z})^{\alpha-2m_1} dm(z).$$

By the condition (4.3) we see that $\alpha - 2m_r > -1$, thus the above integral is finite (see [3]), namely the function is in the L^2-space. That $\bar{\Delta}_{\underline{\mathbf{m}}}(N(z))$ is in $\mathrm{Ker} D^{m+1}$ follows directly from Lemma 2.5. $\qquad\square$

The action π_ν of G on $L^2(\Omega, \mu_\alpha)$ induces an action of $\mathfrak{g}^{\mathbb{C}}$ on the space of C^∞-functions. We prove next that the function $\bar{\Delta}_{\underline{\mathbf{m}}}(q(z))$ is annihilated by the positive root vectors in $\mathfrak{g}^{\mathbb{C}}$. The element in $\mathfrak{p}$, when viewed as holomorphic vector fields, are of the form $\xi_v = v - Q(z)\bar{v}$; thus when acting on C^∞-functions on Ω induced from the regular action of G, they are

$$(\partial_v - \partial_{Q(z)\bar{v}})f + (\partial_{\bar{v}} - \partial_{Q(\bar{z})v})f$$

¿From this it follow that the element $v \in \mathfrak{p}^+ = V$ acts on C^∞-functions induced from π_ν of G is

(4.11) $$\pi_\nu(v)f = \partial_v f - \partial_{Q(\bar{z})v}f,$$

since the infinitesimal action of $v \in \mathfrak{p}^+$ is a translation and it will not contribute in the determinant factor in (4.1).

To study the action of $\mathfrak{p}^+$ on $\bar{\Delta}_{\underline{\mathbf{m}}}(q(z))$, we calculate first the differentiation of $q(z)$.

Lemma 4.5. *The following differentiation formulas hold*

(4.12) $$\partial_v q(z) = Q(q(z))v, \quad \partial_{\bar{w}} q(z) = B(\bar{z},z)^{-1}\bar{w}.$$

In particular if $\bar{w} = Q(\bar{z})v$,

(4.13) $$\partial_{Q(\bar{z})v} q(z) = B(\bar{z},z)^{-1}Q(\bar{z})v = Q(q(z))v,$$

and

(4.14) $$(\partial_v - \partial_{Q(\bar{z})v})q(z) = 0$$

Proof. We use the addition formulas in [6], Appendix, for the quasi-inverses. As special cases we have

(4.15) $$\bar{z}^{z+tv} = (\bar{z}^z)^{tv} = B(\bar{z}^z, tv)^{-1}(\bar{z}^z - tQ(\bar{z}^z)v),$$

and

(4.16) $(\bar{z} + t\bar{w})^z = \bar{z}^z + B(\bar{z}, z)^{-1} B(t\bar{w}, z^{\bar{z}})^{-1} (t\bar{w} - Q(t\bar{w}) z^{\bar{z}}).$

The first order term in t in (4.15) is easily seen to be

$$D(\bar{z}^z, v)\bar{z}^z - Q(\bar{z}^z)v = Q(\bar{z}^z)v,$$

which proves the first formula in (4.12). Similarly we can calculate the first order
term in (4.16) and prove the second formula; using this formula and

$$B(\bar{z}, z)^{-1} Q(\bar{z}) = Q(\bar{z}^z) = Q(q(z)),$$

we get then (4.13). $\square$

We can thus calculate $\pi_\nu(v)$ on $\bar{\Delta}_{\mathbf{m}}(q(z))$ by using (4.11). In view of (4.14) we have

$$\pi_\nu(v)\bar{\Delta}_{\mathbf{m}}(q(z)) = 0.$$

This, together with Lemma 3.2, implies that

Proposition 4.6. *The vector $\Delta_{\underline{\mathbf{m}}}(q(z))$ under the action of π_ν of $\mathfrak{g}^{\mathbb{C}}$ is annihilated
by the positive root vectors.*

We let $A^{2;\alpha}_{\underline{\mathbf{m}}}(\Omega)$ be the subspace of $L^2(\Omega, \mu_\alpha)$ generated by the function $\bar{\Delta}_{\underline{\mathbf{m}}}(q(z))$,
for $\underline{\mathbf{m}}$ given by (4.1). Thus it is a highest weight representation of G. Now it follows
from Lemma 2.2 that

$$\bar{D}^m(\bar{\Delta}_{\underline{\mathbf{m}}}(q(z))) = m!\bar{\Delta}_{\underline{\mathbf{m}}}$$

The vector $\Delta_{\mathbf{m}}$ is the highest weight vector of the weighted Bergman space $L^2_a(\Omega, S_{\underline{\mathbf{m}}}(V), \mu_\alpha)$,
and $\bar{D}^m$ intertwines the G-action π_ν on $A^{2;\alpha}_{\underline{\mathbf{m}}}(\Omega)$ with that on $L^2_a(\Omega, S_{\underline{\mathbf{m}}}(V), \mu_\alpha)$ (see
(4.2)), by Lemma 2.1. Thus it is a non-zero intertwining operator of the two spaces.
We summarize our results in the following

Theorem 4.7. *The relative discrete series $A^{2;\alpha}_{\underline{\mathbf{m}}}(\Omega)$ is G-equivalent to the weighted
Bergman space $L^2_a(\Omega, V^{\underline{\mathbf{m}}}, \mu_\alpha)$ and the corresponding intertwining operator is given by
$\bar{D}^{|\underline{\mathbf{m}}|}$. The highest weight vector of $A^{2;\alpha}_{\underline{\mathbf{m}}}(\Omega)$ is given by $\Delta_{\mathbf{m}}(q(z))$. In particular, the
space $A^{2;\alpha}_{\underline{\mathbf{m}}}(\Omega)$ consists of nearly holomorphic functions.*

Remark 4.8. By the results of Shimeno [11] we see that all the relative discrete
series are obtained in this way.

Remark 4.9. When Ω is of tube type and when $\underline{\mathbf{m}} = m\underline{\mathbf{1}}^r = (m, m, \ldots, m)$, the
above is also proved in [1] by considering the tensor products of Bergman spaces
holomorphic functions with polynomial space of anti-holomorphic functions, and in
[7] by Capelli identity.

5. An example: The case of the unit ball in $\mathbb{C}^n$

In this section we consider the example of the unit ball in $V = \mathbb{C}^n$. The rank $r = 1$, $h(z) = 1 - |z|^2$, and symmetric tensor $S_m(V)$ is itself irreducible under K. We study the adjoint operator $\bar{D}^*$ of $\bar{D}$ instead of $\bar{D}$. The operator $(\bar{D}^*)^m$ is thus an intertwining operator from the weighted Bergman space $L_a^2(\Omega, S_m(V), \mu_\alpha)$ of vector-valued holomorphic functions into the relative discrete series $A_m^{2,\alpha}(\Omega)$. Now the $L_a^2(\Omega, S_m(V), \mu_\alpha)$ has highest weight vector $\otimes^m e_1$. Thus $(\bar{D}^*)^m(\otimes^m e_1)$ is the highest weight vector in $A_m^{2,\alpha}$. We calculate directly here this vector.

Let $D = \bar{D}^*$. It has the following expression on a function f with values in $\otimes^m V$:

$$Df = h(z)^{-\alpha} \otimes^{m-1} B(z, \bar{z}) \operatorname{Tr} \partial \left[h(z)^\alpha (I \otimes \otimes^{m-1} B(z, \bar{z})^{-1}) f \right].$$

To explain the formula we note that, the operator ∂ acting on a $\otimes^m V$-valued function gives a functions with values in $V' \otimes (\otimes^m V) = (V' \otimes V) \otimes (\otimes^{m-1} V)$; the operator Tr is the bilinear pairing between the first factor $V' \otimes V$. Recall that the Bergman operator on the unit ball is

$$B(z, \bar{z}) = (1 - |z|^2)(1 - z \otimes z^*),$$

where $z \otimes z^*$ is the rank one operator on V, $z \otimes z^*(v) = \langle v, z \rangle z$; see [9]. Take $f = \otimes^m e_1$. The above formula then reads

$$D \otimes^m e_1$$
$$= h(z)^{-\alpha-(m-1)} \otimes^{m-1} (1 - z \otimes z^*) \operatorname{Tr} \partial \left[h(z)^{\alpha-2(m-1)}(e_1 \otimes \otimes^{m-1}((1 - |z|^2)e_1 + \bar{z}_1 z)) \right]$$

Performing the differentiation using the Leibniz rule we first differentiate the term $h(z)^{\alpha-2(m-1)}$, and get

$$(5.1) \quad (2(m-1) - \alpha)h(z)^{\alpha-2(m-1)-1}\left(\sum_j \bar{z}_j dz_j\right) \otimes (e_1 \otimes \otimes^{m-1}((1 - |z|^2)e_1 + \bar{z}_1 z).$$

Taking the trace Tr, it is

$$(2(m-1) - \alpha)(1 - |z|^2)^{-1}\bar{z}_1 \otimes^{m-1} e_1.$$

Next we differentiate each factor $(1 - |z|^2)e_1 + \bar{z}_1 z$ in the tensor, and get

$$(5.2) \qquad \qquad \left(-\sum_j \bar{z}_j dz_j\right) \otimes e_1 + \bar{z}_1 \sum_j dz_j \otimes e_j.$$

We perform the operation Tr and observe that each term is vanishing:

$$(5.3) \qquad \operatorname{Tr} e_1 \otimes \left(\left(-\sum_j \bar{z}_j dz_j\right) \otimes e_1 + \bar{z}_1 \sum_j dz_j \otimes e_j\right) = 0.$$

Thus only the first differentiation contributes to the final result, that is

$$D(\otimes^m e_1) = (2(m-1) - \alpha)(1 - |z|^2)^{-1}\bar{z}_1 \otimes^{m-1} e_1.$$

By induction we get

$$D^m e_1^m = C(1 - |z|^2)^{-m} \bar{z}_1^m$$

where

$$C = \prod_{l=0}^{m-1} (2(m-1-j) - \alpha + j).$$

The function $(1 - |z|^2)^{-m}\bar{z}_1^m$ is in $L^2(\Omega, \mu_\alpha)$ if and only if $0 \leq m < \frac{\alpha+1}{2}$. In that case $D^m(\otimes^m e_1)$ is a non-zero multiple of $(1 - |z|^2)^{-m}\bar{z}_1^m$. The quasi-inverse is $q(z) = (1 - |z|^2)^{-1}\bar{z}$, and the vector constructed in Theorem 4.7 is $[e_1^m, \otimes q(z)] = (1 - |z|^2)^{-m}\bar{z}_1^m$, and thus the two methods give the same result.

One might also in the beginning work with the operator $D = -(\bar{D})^*$ instead of $\bar{D}$. However we note that for a general bounded symmetric domain the formula for the operator D is much more involved.

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